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## Definitions

1. Define  $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ .

$$= \left\{ \vec{b} \text{ in } \mathbb{R}^m : \vec{b} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 \right. \\ \left. \text{for some } c_1, c_2, c_3 \text{ in } \mathbb{R} \right\}$$

= the set of all  $\vec{b}$  that can be gotten by scaling & adding  $\vec{v}_1, \vec{v}_2, \vec{v}_3$

2. Define linear Independence of vectors  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ .

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is linearly independent  
 $\Leftrightarrow$  the equation  $x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 = \vec{0}$   
 has ONLY the trivial solution

3. Define " $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear transformation"

$$\left. \begin{array}{l} T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \\ \text{AND} \\ T(c \cdot \vec{u}) = c \cdot T(\vec{u}) \end{array} \right\} \begin{array}{l} \text{for ALL } \vec{u}, \vec{v} \text{ in } \mathbb{R}^n \\ \text{and ALL } c \text{ in } \mathbb{R}. \end{array}$$

4. Define " $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is one-to-one"

$T(\vec{x}) = \vec{b}$  has AT MOST one solution  
 for each  $\vec{b}$  in  $\mathbb{R}^m$

5. Define " $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is onto"

$T(\vec{x}) = \vec{b}$  has AT LEAST one solution  
 for each  $\vec{b}$  in  $\mathbb{R}^m$

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## General Linear Transformations

1. Let  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be defined so that  $T(\vec{x}) = A\vec{x}$  where

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 2 & 3 \\ 1 & -4 & -1 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{c} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

(a) Compute  $T(\vec{u})$ . Is  $\vec{u}$  a solution to  $T(\vec{u}) = \vec{b}$ ?

Plugging it in!

$$T(\vec{u}) = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 2 & 3 \\ 1 & -4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix} \neq \vec{b}$$

 $\vec{u}$  is NOT a solution to  $T(\vec{u}) = \vec{b}$ (b) Find all solutions to the equation  $T(\vec{x}) = \vec{b}$  solve  $A\vec{x} = \vec{b}$ 

$$\text{Solve } \left[ \begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ -1 & 2 & 3 & 2 \\ 1 & -4 & -1 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array} \right] \begin{array}{l} r_2 + r_1 \\ r_3 - r_1 \end{array}$$

$$\sim \left[ \begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & -5 & -4 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} r_1 - 3r_2 \end{array}$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 0 & -5 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] \Leftrightarrow \begin{cases} x_1 - 5x_3 = -4 \\ x_2 - x_3 = -1 \\ x_3 \text{ free} \end{cases} \Leftrightarrow \begin{cases} x_1 = -4 + 5x_3 \\ x_2 = -1 + x_3 \\ x_3 \text{ free} \end{cases}$$

(c) Is  $\vec{c}$  in the range of  $T$ ? Justify your answer. solve  $A\vec{x} = \vec{c}$ 

$$\left[ \begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ -1 & 2 & 3 & 2 \\ 1 & -4 & -1 & 5 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 6 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 5 \end{array} \right]$$

solutions are

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 + 5x_3 \\ -1 + x_3 \\ 0 + x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$$

System is inconsistent  $\Rightarrow$  NO solution.

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2. (No Computation) Define  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be defined so that  $T(\vec{x}) = A\vec{x}$  where

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ -3 & 2 \end{bmatrix}$$

- (a) What is the domain of  $T$ ? *( $A\vec{x}$  defined when  $\vec{x}$  has 2 rows)*

$$\text{domain} = \mathbb{R}^2$$

- (b) What is the co-domain of  $T$ ?

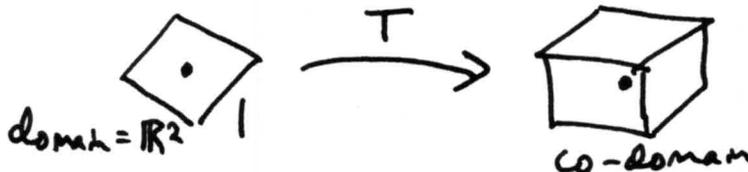
*(scaling + adding across produces vectors w/ 3 rows)*

$$\text{co-domain} = \mathbb{R}^3$$

- (c) Describe the Range of  $T$  as the span of a set of vectors.

$$\begin{aligned} \text{Range}(T) &= \{ \vec{b} \in \mathbb{R}^3 : T(\vec{x}) = \vec{b} \text{ for some } \vec{x} \in \mathbb{R}^2 \} \\ &= \{ \vec{b} \in \mathbb{R}^3 : x_1 \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \vec{b} \text{ for } x_1, x_2 \in \mathbb{R} \} \\ &= \text{Span} \left\{ \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\} \end{aligned}$$

- (d) Sketch the general form of  $T$  as a function from the domain to the co-domain. Use a square to represent  $\mathbb{R}^2$  and a cube to represent  $\mathbb{R}^3$ .



3. (No Computation) How many rows and columns must a matrix  $A$  have in order to define a mapping from  $\mathbb{R}^5$  into  $\mathbb{R}^7$  by the rule  $T(\vec{x}) = A\vec{x}$ ?

inputs  
# columns      outputs  
# rows

$$A \text{ is } 7 \times 5$$

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4. Let  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ,  $\vec{v}_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ , and  $\vec{v}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ , and let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear transformation that maps  $\vec{x}$  to  $x_1\vec{v}_1 + x_2\vec{v}_2$ . Find a matrix  $A$  so that  $T(\vec{x}) = A\vec{x}$  for every  $\vec{x}$ .

$$\begin{aligned} T(\vec{x}) &= x_1\vec{v}_1 + x_2\vec{v}_2 \\ &= x_1 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 2 & 3 \\ -5 & -2 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

5. Find the standard matrix of the linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that sends  $\vec{e}_1$  to  $\vec{e}_1 - 3\vec{e}_2$  and leaves  $\vec{e}_2$  unchanged.

$$T(\vec{e}_1) = \vec{e}_1 - 3\vec{e}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$T(\vec{e}_2) = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

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6. Suppose that  $A$  and  $B$  are  $m \times n$  matrices, and that  $\vec{x} \in \mathbb{R}^n$ .

Prove that  $(A+B)\vec{x} = A\vec{x} + B\vec{x}$  from first principles.

(using only definitions  
& computations  
(no theorems).)

$$\text{Let } A = [\vec{a}_1 \ \dots \ \vec{a}_n]$$

$$B = [\vec{b}_1 \ \dots \ \vec{b}_n]$$

$$\text{and } \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$(A+B)\vec{x} = [(\vec{a}_1 + \vec{b}_1) \ \dots \ (\vec{a}_n + \vec{b}_n)] \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1(\vec{a}_1 + \vec{b}_1) + \dots + x_n(\vec{a}_n + \vec{b}_n)$$

$$= \boxed{x_1\vec{a}_1 + x_1\vec{b}_1 + \dots + x_n\vec{a}_n} + x_n\vec{b}_n$$

$$= [\vec{a}_1 \ \dots \ \vec{a}_n] \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + [\vec{b}_1 \ \dots \ \vec{b}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= A\vec{x} + B\vec{x}$$

$$\text{So } (A+B)\vec{x} = A\vec{x} + B\vec{x}$$

QED.

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6. Prove that  $T(\vec{x}) = \underline{3\vec{x}}$  is a linear transformation.

$P \leftrightarrow Q$

$T(\vec{x})$  is linear  $\Leftrightarrow T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$   
 AND  
 $T(c\vec{u}) = c \cdot T(\vec{u})$  for all  $c \in \mathbb{R}$   
 and all  $\vec{u}, \vec{v}$

Compute

①  $T(\vec{u} + \vec{v}) = 3 \cdot (\vec{u} + \vec{v}) = 3\vec{u} + 3\vec{v} = T(\vec{u}) + T(\vec{v})$  ✓

②  $T(c\vec{u}) = 3 \cdot (c\vec{u}) = c \cdot 3\vec{u} = c \cdot T(\vec{u})$  ✓

---

therefore  
 $T$  is linear

7. Prove that  $T(\vec{x}) = \underline{3\vec{x} + 1}$  is not linear transformation, where  $\vec{1} = [1]$ .

$P \leftrightarrow Q$

$T(\vec{x})$  is linear  $\Leftrightarrow T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$   
 AND  
 $T(c\vec{u}) = c \cdot T(\vec{u})$  for all  $c \in \mathbb{R}$   
 and all  $\vec{u}, \vec{v}$

Counter example

pick  $\vec{u} = [1]$  and  $\vec{v} = [2]$

$T(\vec{u} + \vec{v}) = T([1] + [2]) = T([3]) = [3 \cdot 3 + 1] = [10]$  ✗

BUT

$T(\vec{u}) + T(\vec{v}) = T([1]) + T([2]) = \underbrace{[3 \cdot 1 + 1]}_4 + \underbrace{[3 \cdot 2 + 1]}_7 = [11]$

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therefore  
 $T$  is NOT linear

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8. Let transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a linear transformation with

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } T(\vec{e}_2) = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}.$$

(a) Find the standard matrix  $A$  of  $T$ .

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 1 & 0 \end{bmatrix}$$

General form of  $T$



(b) Determine if the transformation  $T$  is one-to-one.

$T$  one-to-one  $\Leftrightarrow$  columns of  $A$  are independent.

Count solutions to  $A\vec{x} = \vec{0}$

$$\left[ \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

↑ ↑  
pivot in each column  
 $\Rightarrow$   
unique solution  
 $\Rightarrow$   
 $A$  is indep

$T$  is one-to-one.

(c) Determine if the transformation  $T$  is onto.

$T$  is onto  $\Leftrightarrow$  columns of  $A$  span  $\mathbb{R}^3$

$\Leftrightarrow$  pivot in each row.

BUT no pivot in Row 3

So  $T$  is not onto.

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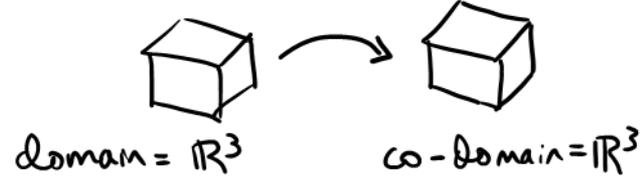
9. Let transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a linear transformation with

$$T(\vec{e}_1) = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, T(\vec{e}_2) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \text{ and } T(\vec{e}_3) = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}.$$

(a) Find the standard matrix  $A$  of  $T$ .

$$A = \begin{bmatrix} 2 & 2 & 0 \\ -2 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

General form of  $T$



(b) Determine if the transformation  $T$  is one-to-one.

$T$  is one-to-one  $\Leftrightarrow$  columns of  $A$  are independent

$$\left[ \begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ -2 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\uparrow$   
no pivot in col 3  $\Rightarrow A$  not indep  $\Rightarrow$   $\nexists$  NOT one-to-one

(c) Determine if the transformation  $T$  is onto.

$T$  is onto  $\Leftrightarrow$  columns of  $A$  span  $\mathbb{R}^3$   
 $\Leftrightarrow$  pivot in each row.

But

no pivot in row 3

$\Rightarrow T$  NOT onto.

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10. Suppose that  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is a linear transformation with standard form matrix  $A$ .  
 Prove that  $T$  is *not* onto. (Cite all relevant definitions and theorems by number).

$\leftarrow$   $A$  is  $3 \times 2$   $\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$

$P \Leftrightarrow Q$   $T$  is onto  $\Leftrightarrow$  columns of  $A$  span  $\mathbb{R}^3$   
 $\Leftrightarrow A$  has pivot in each row

$\neg Q$  But you cannot fit 3 pivots into 2 columns

$\neg P$  So  $T$  cannot be onto.

11. Give an example of a linear transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  that is one-to-one (Hint: define  $T$  by choosing its standard matrix).

let  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$  then  $T(\vec{x}) = A\vec{x}$  is one-to-one.

12. Suppose  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is linear. Is it true that  $T$  is one-to-one if and only if  $T$  is onto? Why doesn't this violate the invertible matrix theorem?

NO. But the invertible matrix theorem ONLY applies to square matrices.

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←  $A$  is  $2 \times 3$   $\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$

13. Suppose  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is linear. Prove that  $T$  cannot be one-to-one.

$P \leftrightarrow Q$   $T$  is one-to-one  $\Leftrightarrow$  the standard matrix  $A$  has independent columns  
 $\Leftrightarrow A$  has a pivot in each column.

$\neg Q$  But you cannot fit 3 pivots into 2 rows

---

$\neg P$  So  $T$  cannot be one-to-one

14. Give an example of a linear transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  that is onto (Hint: define  $T$  by choosing its standard matrix).

$$\text{let } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

then  $T(\vec{x}) = A\vec{x}$  is onto.

15. Suppose  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is linear. Is it true that  $T$  is one-to-one if and only if  $T$  is onto? Why doesn't this violate the invertible matrix theorem?

No.

Because the IMT ONLY applies to square matrices.

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## Matrix Operations

1. Compute the following matrix operations, or explain why they are undefined.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 3 & -2 \\ 5 & 1 \end{bmatrix}$$

(a)  $AB$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}_{2 \times 3} \begin{bmatrix} 0 & 1 \\ 3 & -2 \\ 5 & 1 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 5 & 3 \\ -1 & 8 \end{bmatrix}_{2 \times 2}$$

(b)  $BA$

$$= \begin{bmatrix} 0 & 1 \\ 3 & -2 \\ 5 & 1 \end{bmatrix}_{3 \times 2} \begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}_{2 \times 3} = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & 1 \\ 13 & -2 & 6 \end{bmatrix}_{3 \times 3}$$

(c)  $A^T$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 \\ 0 & -2 \\ 1 & 1 \end{bmatrix}$$

(d)  $2A - 3B$

$$2 \begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix} - 3 \begin{bmatrix} 0 & 1 \\ 3 & -2 \\ 5 & 1 \end{bmatrix} \quad \text{DNE (dimensions don't match)}$$

(e)  $2A^T - 3B$

$$2 \begin{bmatrix} 2 & 3 \\ 0 & -2 \\ 1 & 1 \end{bmatrix} - 3 \begin{bmatrix} 0 & 1 \\ 3 & -2 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 0 & -4 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ -9 & 6 \\ -15 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -9 & 2 \\ -13 & -1 \end{bmatrix}$$

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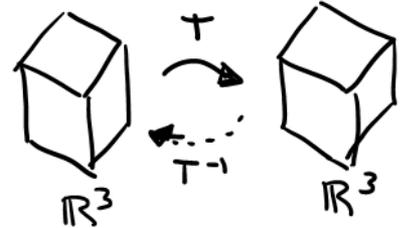
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2. Let

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix}_{3 \times 3}$$

(a) State the domain and co-domain of the transformation  $T$  defined by  $T(\vec{x}) = A\vec{x}$ .

domain = input space =  $\mathbb{R}^3$   
 co domain = output space =  $\mathbb{R}^3$

(b) Compute  $A^{-1}$ .

$$\left[ \begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & -1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} r_2 - r_1 \\ r_3 - r_1 \end{array}$$

$$\sim \left[ \begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -2 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} r_2 - 2r_3 \\ r_3 - 2r_3 \end{array} \sim \left[ \begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array} \right]$$

$$\begin{array}{l} 1 - 5/2 \\ 3/2 - 5/2 = -3/2 \end{array} \left| \sim \left[ \begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & 0 & 5/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array} \right] \begin{array}{l} r_1 - 5r_2 \\ \underbrace{\hspace{10em}}_{A^{-1}} \end{array}$$

(c) Let  $\vec{b} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ . Use the inverse of  $A$  to solve the matrix equation  $A\vec{x} = \vec{b}$ .

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \\ \Leftrightarrow A^{-1}A\vec{x} &= A^{-1} \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \\ \mathbb{I}_3 \Leftrightarrow \vec{x} &= A^{-1} \cdot \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -3/2 & 0 & 5/2 \\ 1/2 & 0 & -1/2 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} \end{aligned}$$

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3. Rewrite  $(AB)^{-1}(B+A)$  using the properties of matrix operations.

(Careful: Multiplication is *not* commutative).

$$\begin{aligned}
 & \overbrace{(AB)^{-1}(B+A)} \\
 &= (AB)^{-1}B + (AB)^{-1}A \quad \text{RIGHT distribute} \\
 &= \underbrace{B^{-1}A^{-1}}B + B^{-1}\underbrace{A^{-1}A} \\
 &= B^{-1}A^{-1}B + B^{-1}I_n \\
 &= B^{-1}A^{-1}B + B^{-1} \quad \leftarrow \text{cannot rewrite}
 \end{aligned}$$

4. Rewrite  $(B+A)(AB)^{-1}$  using the properties of matrix operations.

(Careful: Multiplication is *not* commutative).

$$\begin{aligned}
 & \overbrace{(B+A)(AB)^{-1}} \\
 &= B(AB)^{-1} + A(AB)^{-1} \quad \text{left distribute} \\
 &= B \cdot \underbrace{B^{-1}A^{-1}} + A \cdot \underbrace{B^{-1}A^{-1}} \\
 &= I_n \cdot A^{-1} + AB^{-1}A^{-1} \\
 &= A^{-1} + AB^{-1}A^{-1} \quad \leftarrow \text{cannot rewrite}
 \end{aligned}$$

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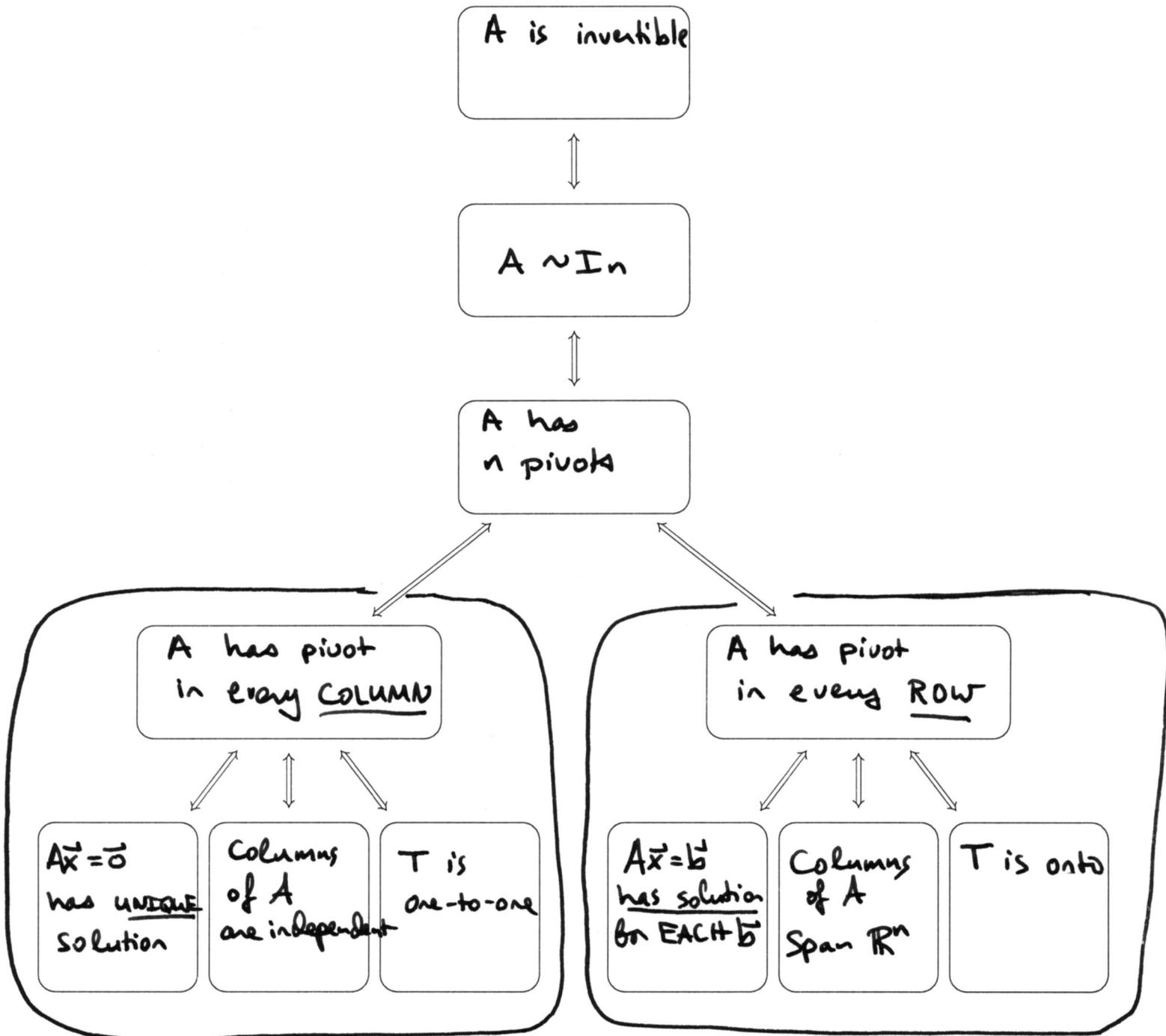
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## Square Matrices

### The Invertible Matrix Theorem (IMT)

Suppose that  $A$  is an  $n \times n$  matrix, and that  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is defined by  $T(\vec{x}) = A\vec{x}$ .

Then, the following are equivalent.



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1. Suppose that  $A$  is an  $n \times n$  matrix. Prove the following statements by "walking through the tree" of the Invertible Matrix Theorem. **You must show every step.**

- (a) Suppose that an  $n \times n$  matrix  $A$  is invertible.  
Prove that the columns of  $A$  span  $\mathbb{R}^n$ .

If  $A$  is invertible then  $A \sim I_n$

so  $A$  has  $n$  pivots

so  $A$  has pivot in each Row

Therefore, the columns of  $A$  span  $\mathbb{R}^n$

- (b) Suppose that an  $n \times n$  matrix  $A$  is not invertible. Prove that the columns of  $A$  are linearly dependent.

If  $A$  is NOT invertible, then  $A \not\sim I_n$

so  $A$  doesn't have  $n$  pivots

so doesn't have pivot in each column

Thus, the columns of  $A$  are dependent

- (c) Suppose that  $A$  is an  $n \times n$  matrix, and that  $A\vec{x} = \vec{0}$  has a unique solution. Prove that  $A$  is invertible.

If  $A\vec{x} = \vec{0}$  has a unique solution

then  $A$  has pivot in each column

so  $A$  has  $n$  pivots

so  $A \sim I_n$

Therefore,  $A$  is invertible

- (d) Suppose that  $A$  is an  $n \times n$  matrix, and that  $A\vec{x} = \vec{b}$  does not have a solution for some  $\vec{b}$  in  $\mathbb{R}^n$ . Prove that  $A$  is not invertible.

If  $A\vec{x} = \vec{b}$  doesn't always have a solution

then  $A$  doesn't have a pivot in each row

so  $A$  doesn't have  $n$  pivots

so  $A \not\sim I_n$

thus,  $A$  is not invertible.

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2. Determine if the following matrices are invertible using as few computations as possible.

Square ✓

(a)  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

NOTE: Neither column is a multiple of other  
AND  $n=2$

$\Rightarrow$  columns are independent

$\Rightarrow A$  is invertible

Recall

$(n=2) \Rightarrow$  (Indep  $\Leftrightarrow$  neither column is a multiple of the other)

Square ✓

(b)  $A = \begin{bmatrix} -2 & 6 \\ 1 & -3 \end{bmatrix}$

NOTE:  $\begin{bmatrix} 6 \\ -3 \end{bmatrix} = (-3) \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

and  $n=2$

$\Rightarrow$  columns are Dependent

$\Rightarrow A$  is NOT invertible

Square ✓

(c)  $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

NOTE:  $A \sim I_3$

SO  $A$  is invertible

Square ✓

(d)  $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

$\uparrow$   
 no pivot in 2nd column

$\Rightarrow$   
doesn't have 3 pivots

$\Rightarrow$   
 $A$  is NOT invertible.

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## Theorems

**Theorem 2** There are 3 cases for the reduced echelon form of a linear system's augmented matrix

1. The system has 0 solutions if it contains  $[0 \dots 0 | \text{nonzero}]$
2. The system has 1 solutions if it has pivot in each coeff. COLUMN
3. The system has  $\infty$ -many solutions if some coeff. COLUMN lacks a pivot

**Theorem 4:** The columns of an  $m \times n$  matrix  $A$  span  $\mathbb{R}^m$

if and only if there is a pivot in every ROW.

## Shortcuts to Recognize Dependence

- If one column of  $A$  is a multiple of another, then the columns of  $A$  are linearly dependent.
- If  $\{\vec{a}_1, \dots, \vec{a}_n\}$  contains  $\vec{0}$ , then  $\{\vec{a}_1, \dots, \vec{a}_n\}$  is linearly dependent.
- If an  $m \times n$  matrix  $A$  has more columns than rows (if  $n > m$ ), then the columns of  $A$  are linearly dependent.

**Theorem 5** If  $A$  is an  $m \times n$  matrix,  $\vec{u}, \vec{v} \in \mathbb{R}^n$  and  $c \in \mathbb{R}$ , Then

- $A(\vec{u} + \vec{v}) = \underline{A\vec{u} + A\vec{v}}$
- $A(c \cdot \vec{u}) = \underline{c \cdot A\vec{u}}$

## Properties of Linear Transformations

- If  $T$  is linear, then  $T(\vec{0}) = \underline{\vec{0}}$
- $T$  is linear  $\iff T(c \cdot \vec{u} + d \cdot \vec{v}) = \underline{c \cdot T(\vec{u}) + d \cdot T(\vec{v})}$

**Theorem 10** Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be linear.

Then there is a unique  $m \times n$  matrix  $A$  s.t.  $T(\vec{x}) = A\vec{x}$ .

In Fact,  $A = \underline{\begin{bmatrix} T(\vec{e}_1) & \dots & T(\vec{e}_n) \end{bmatrix}}$

**Theorem 12** Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be linear with standard matrix  $A$ . T

- (a)  $T$  is onto  $\iff$  the columns of  $A$  span  $\mathbb{R}^m$   $\iff$   $A$  has pivot in each ROW
- (b)  $T$  is one-to-one  $\iff$  the columns of  $A$  are independent  $\iff$   $A$  has pivot in each COLUMN

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## Determinants

1. Let  $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ . Compute  $\det(A)$ .

$$\det(A) = 2 \cdot 4 - 1 \cdot 3 = 8 - 3 = 5$$

2. Let  $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ . Compute  $\det(A)$ .

$$\begin{aligned} \det(A) &= 1 \cdot \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} \\ &= 1 \cdot (1 \cdot 1 - 2 \cdot (-1)) - 2(3 \cdot 1 - 2 \cdot 1) + 3 \cdot (-1 \cdot 3 - 1 \cdot 1) \\ &= (1 + 2) - 2(1) + 3(-3 - 1) \\ &= 3 - 2 - 12 = -11 \end{aligned}$$

3. Let  $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 5 \\ 2 & 6 & 0 \end{bmatrix}$ . Compute  $\det(A)$ .

$$\begin{aligned} \det A &= 0 \cdot \begin{vmatrix} 0 & 5 \\ 6 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 5 \\ 2 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix} \\ &= 0 - 1(0 - 10) + 2(6 - 0) \\ &= 0 - 1(-10) + 2 \cdot 6 \\ &= 10 + 12 \\ &= 22 \end{aligned}$$

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4. Find  $k$  so that  $A = \begin{bmatrix} 1 & k & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 3 \end{bmatrix}$  is invertible.

$$A \text{ invertible} \Leftrightarrow \det A \neq 0$$

$$\begin{aligned} \det A &= 1 \cdot \begin{vmatrix} 0 & 1 \\ -1 & 3 \end{vmatrix} - k \cdot \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} \\ &= 1(0 - (-1)) - k(6 - 1) + 0 \cdot (-2 - 0) \\ &= 1 \quad -k(5) \quad + 0 \\ &= 1 - 5k. \end{aligned}$$

$$\begin{aligned} \text{So } A \text{ invertible} &\Leftrightarrow 0 \neq 1 - 5k \\ &\Leftrightarrow k \neq \frac{1}{5}. \end{aligned}$$

5. Find  $k$  so that  $A = \begin{bmatrix} 0 & k & 2 \\ 3 & 0 & k \\ 1 & k & 0 \end{bmatrix}$  is invertible.

$$A \text{ invertible} \Leftrightarrow \det A \neq 0$$

$$\begin{aligned} \det A &= 0 \cdot \begin{vmatrix} 0 & k \\ k & 0 \end{vmatrix} - k \begin{vmatrix} 3 & k \\ 1 & 0 \end{vmatrix} + 2 \begin{vmatrix} 3 & 0 \\ 1 & k \end{vmatrix} \\ &= 0(0 - k^2) - k(0 - k) + 2(3k - 0) \\ &= k^2 + 6k \end{aligned}$$

$$\begin{aligned} A \text{ invertible} &\Leftrightarrow 0 \neq k^2 + 6k \\ &\Leftrightarrow 0 \neq k(k+6) \\ &\Leftrightarrow \text{BOTH } k \neq 0 \text{ AND } k \neq -6. \end{aligned}$$

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4. Let  $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$ . Compute  $\det(A)$ .

$$\det(A) = 0 \cdot \underbrace{\begin{vmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}}_0 - 1 \cdot \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{vmatrix} + 0 \cdot \underbrace{\begin{vmatrix} 25 & 0 \\ 0 & 0 \\ 1 & 0 & 3 \end{vmatrix}}_0 - 0 \cdot \underbrace{\begin{vmatrix} 25 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix}}_0$$

$$= -1 \left( 2 \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} - \underbrace{0 \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix}}_0 + 0 \cdot \underbrace{\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}}_0 \right)$$

$$= (-1)(2)(2 \cdot 3 - 0 \cdot 0)$$

$$= -12$$

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5. You can compute the determinant of a large matrix by *reducing to echelon form* using only interchange and replacement, tracking how the determinant changes at each step.

(a) Interchange: flips the sign of the determinant

$$\det \begin{bmatrix} \text{--- } r_1 \text{ ---} \\ \text{--- } r_2 \text{ ---} \\ \text{--- } r_3 \text{ ---} \end{bmatrix} = (-1) \cdot \det \begin{bmatrix} \text{--- } r_2 \text{ ---} \\ \text{--- } r_1 \text{ ---} \\ \text{--- } r_3 \text{ ---} \end{bmatrix} \quad \curvearrowright$$

(b) Replacement: doesn't change sign of the determinant.

$$\det \begin{bmatrix} \text{--- } r_1 \text{ ---} \\ \text{--- } r_2 \text{ ---} \\ \text{--- } r_3 \text{ ---} \end{bmatrix} = \det \begin{bmatrix} \text{--- } r_1 \text{ ---} \\ \text{--- } r_2 + 3r_1 \text{ ---} \\ \text{--- } r_3 \text{ ---} \end{bmatrix} \quad r_2^* = r_2 + 3r_1$$

(c) Rescaling: If you can factor a const. out of a row, you can pull it through the determinant

$$\det \begin{bmatrix} \text{--- } r_1 \text{ ---} \\ \text{--- } 2 \cdot r_2 \text{ ---} \\ \text{--- } r_3 \text{ ---} \end{bmatrix} = (2) \cdot \det \begin{bmatrix} \text{--- } r_1 \text{ ---} \\ \text{--- } r_2 \text{ ---} \\ \text{--- } r_3 \text{ ---} \end{bmatrix} \quad r_2^* = \frac{1}{2} r_2$$

6. Let  $A = \begin{bmatrix} 1 & 4 & 2 & 4 \\ 2 & 8 & 4 & 10 \\ 0 & 4 & 6 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix}$ . Compute  $\det(A)$ .

$$\det A = \det \begin{bmatrix} 1 & 4 & 2 & 4 \\ 0 & 0 & 0 & 2 \\ 0 & 4 & 6 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix} \quad \text{replacement}$$

$$= (-1) \det \begin{bmatrix} 1 & 4 & 2 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 4 & 6 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{interchange}$$

$$= (-1) \cdot \det \begin{bmatrix} 1 & 4 & 2 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{replacement}$$

$$= (-1) \cdot 1 \cdot 2 \cdot 2 \cdot 2$$

$$= -8$$

determinant of a triangular matrix.